JGP - Vol. 7, n. 2, 1990

The twistor equation on Riemannian manifolds

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Abstract. *It is shown that a twistor spinor on a Riemannian manifold defines a conformal deformation to an Einstein manifold. Twistor spinors on 4-manifolds are considered. A characterisation of the hyperbolic space is given. Moreover the solutions of the twistor equation on warped products $M^n \times \mathbb{R}$, where M^n is an Einstein manifold, are described.*

We study n-dimensional Riemannian spin manifolds (M^n, g), $n \geqslant 3$, admitting non-trivial twistor spinors. A twistor spinor is a spinor field $\psi \in \Gamma(S)$ satisfying the differential equation

$$\nabla^S_X \psi + \frac{1}{n} X \cdot D\psi = 0$$

for all vector fields X.

The present paper is related to investigations by Th. Friedrich ([3]) and A. Lichnerowicz ([7, 8]).

In the first section we introduce some notations and give a short summary of previous results.

In Section 2 we show that a Riemannian spin manifold, which admits a solution ψ of the twistor equation satisfying $C^2_\psi + Q_\psi > 0$, is conformally equivalent to an Einstein manifold with positive scalar curvature, where any twistor

1980 MSC: 32 L 25 53 C 20
Key words: Twistors, Riemannian Manifolds.

spinor is conformally equivalent to the sum of two real Killing spinors. On the other hand, for a Riemannian spin manifold M^n with a twistor spinor ψ satisfying $C_\psi = Q_\psi = 0$ we obtain that $M^n \setminus N_\psi$ is conformally equivalent to a Ricci-flat space and $\psi|_{M^n \setminus N_\psi}$ becomes a parallel spinor field. Here C_ψ and Q_ψ are real constants depending on ψ and N_ψ denotes the zero set of ψ. We note that the result for the second case can also be deduced from Proposition 6 in [3]. Similar results can be found in papers of A. Lichnerowicz concerning twistor spinors (see [9, 10]).

In Section 3 we study twistor spinors on 4-manifolds and give informations concerning the dimension of the kernel of the twistor operator $\mathcal{D}$ on connected and simply connected Riemannian 4-manifolds.

In Section 4 we prove

PROPOSITION. *Let (M^n, g), $n \geqslant 3$, be a complete connected spin manifold. Furthermore, let (M^n, g) be an Einstein manifold with non-positive scalar curvature $R \leqslant 0$. Suppose that $\psi \not\equiv 0$ is a non-parallel twistor spinor on M^n such that the function $f : M^n \to [0, \infty)$ defined by $f(x) = (\psi(x), \psi(x))$, $x \in M^n$, attains a minimum.*

Then

(i) If $R < 0$, then (M^n, g) is isometric to the hyperbolic space with sectional curvature $R/(n(n-1))$.

(ii) If $R = 0$, then (M^n, g) is isometric to the space $\mathbb{R}^n$ with the standard metric.

In the last section we describe the solutions of the twistor equation on the warped product $(M \times \mathbb{R}, f^2(t)\, g \oplus dt^2)$ for an Einstein manifold (M, g) and a function $f : \mathbb{R} \to (0, \infty)$.

Furthermore, we give examples of warped products admitting twistor spinors with an arbitrary number of zeros.

The author thanks Th. Friedrich for introducing to the subject and helpful comments.

1. NOTATIONS AND PREVIOUS RESULTS

Let (M^n, g) be a n-dimensional Riemannian spin manifold, $n \geqslant 3$, and let S be the spinor bundle of (M^n, g) equipped with the standard hermitian inner product $\langle\ ,\ \rangle$. Denote by ∇^S the covariant derivative on the spinor bundle induced by the Levi-Civita connection ∇ on M^n.

A twistor spinor on (M^n, g) is a spinor field $\psi \in \Gamma(S)$ solving the differential equation

$$\nabla^S_X \psi + \frac{1}{n} X \cdot D\psi = 0$$

for all vector fields $X \in \Gamma(TM^n)$, where D denotes the Dirac operator and $X \cdot \varphi$ expresses the Clifford multiplication of the vector field X by the spinor field φ. It is well-known that a spinor field $\psi \in \Gamma(S)$ is a twistor spinor if and only if the expression $X \cdot \nabla^S_X \psi$ does not depend on the vector field X, where $|X| \equiv 1$. A spinor field $\psi \in \Gamma(S)$ satisfying the differential equation

$$\nabla^S_X \psi = \lambda X \cdot \psi$$

for all $X \in \Gamma(TM^n)$, where $\lambda \in \mathbb{C}$, is called Killing spinor. Any Killing spinor is a twistor spinor.

The twistor equation characterizes the kernel Ker $\mathcal{D}$ of the twistor operator $\mathcal{D}$. $\mathcal{D}$ is a conformally invariant operator, i.e. if $\bar{g} = \lambda g$ is a conformal change of the metric and $^{-} : S \to \bar{S}$ denotes the natural isomorphism of the spin bundles then $\psi \in \text{Ker}\, \mathcal{D}$ if and only if $\lambda^{1/4}\bar{\psi} \in \text{Ker}\, \bar{\mathcal{D}}$. In addition to the conformal invariance of the operator $\mathcal{D}$ the existence of non-trivial twistor spinors forces properties of the conformal structure of the manifold. If we consider the Weyl tensor W of the Riemannian manifold as a 2-form with values in the bundle End(S), then we obtain $W\psi = 0$ for any twistor spinor ψ.

On the space Ker $\mathcal{D}$ of all twistor spinors the expression $C_\psi = Re\,\langle D\psi, \psi\rangle$ is an invariant of order two and

$$Q_\psi = |\psi|^2 |D\psi|^2 - C_\psi^2 - \sum_{j=1}^{n} (Re\,\langle D\psi, e_j \cdot \psi\rangle)^2 \geqslant 0,$$

where $e_1, \ldots, e_n$ is an orthonormal frame on M^n, is an invariant of order four. In our paper we will essentially use this fact to study twistor spinors.

Further, if $\psi \in \text{Ker}\, \mathcal{D}$ then

$$(1.1) \qquad D^2\psi = \frac{Rn}{4(n-1)}\psi$$

and

$$(1.2) \qquad \nabla^S_X (D\psi) = \frac{n}{2} L(X) \cdot \psi, \qquad X \in \Gamma(TM^n)$$

where L denotes the (1,1)-tensor defined by

$$L(X) = \frac{1}{n-2}\left(\frac{R}{2(n-1)} X - \text{Ric}(X)\right), \qquad X \in TM^n.$$

In the case that the manifold (M^n, g) is an Einstein manifold we have some more informations. It is easy to prove that if (M^n, g) is an Einstein manifold, then $D(Ker\,\mathscr{D}) \subseteq Ker\,\mathscr{D}$.

Moreover, the (1,1)-tensor L is given by $L = R/(2n(n-1))$ id. Finally we remark that $(X \cdot \psi, Y \cdot \varphi) = g(X\,Y) \mid \varphi \mid^2$ for all vector fields X and Y where $(\,,\,)$ denotes the real part of $\langle\,,\,\rangle$.

We refer to [3, 7, 8] for more details.

2. THE CONFORMAL DEFORMATION TO AN EINSTEIN MANIFOLD DEFINED BY A TWISTOR SPINOR

We start our consideration concerning the conformal structure of Riemannian spin manifolds, which admit a non-trivial solution of the twistor equation, with

PROPOSITION 2.1. *Let (M^n, g) be a n-dimensional Riemannian spin manifold, $n \geqslant 3$, with a twistor spinor ψ, $|\psi| \equiv 1$. Then (M^n, g) is an Einstein manifold with non-negative scalar curvature*

$$R = \frac{4(n-1)}{n}(C_\psi^2 + Q_\psi).$$

Proof. We choose a local orthonormal frame $e_1, \ldots, e_n$ on M^n. Since ψ is a twistor spinor, we have

$$\nabla^S_{e_j} \psi = -\frac{1}{n} e_j \cdot D\psi$$

and

$$\nabla^S_{e_j}(D\psi) = \frac{n}{2} L(e_j) \cdot \psi$$

for $j = 1, \ldots, n$.

Consequently

$$(2.1) \qquad \frac{n}{2}\langle L(e_j) \cdot \psi,\ e_i \cdot \psi\rangle = \langle \nabla^S_{e_j}(D\psi), e_i \cdot \psi\rangle =$$

$$= e_j(\langle D\psi, e_i \cdot \psi\rangle) - \langle D\psi, e_i \cdot \nabla^S_{e_j} \psi\rangle$$

Because of $|\psi| \equiv 1$ we have $0 = X\langle \psi, \psi\rangle = 2(\nabla^S_X \psi, \psi) = -2/n\,(X \cdot D\psi, \psi)$ for $X \in \Gamma(TM^n)$.

Hence

(2.2) $(e_j \cdot D\psi, \psi) = 0$ for $j = 1, \dots, n$

The real part of equation (2.1) yields

$$\frac{n}{2} L_{ij} |\psi|^2 = -(D\psi, e_i \cdot \nabla^S_{e_j} \psi) =$$

$$= -\left(e_i \cdot D\psi, \frac{1}{n} e_j \cdot D\psi\right) = -g_{ij} \frac{1}{n} |D\psi|^2,$$

i.e. $L_{ij} = -2/n^2\, g_{ij} |D\psi|^2$.

Equation (2.2) implies $|D\psi|^2 = (D^2\psi, \psi)$, from which $|D\psi|^2 = Rn/(4(n-1))$ follows.

Thus $L_{ij} = -R/(2n(n-1))\, g_{ij}$, which is equivalent to $Ric = R/n\, g$.

Consequently, (M^n, g) is an Einstein manifold and, applying $|D\psi|^2 = Rn/(4(n-1)) \geq 0$, the scalar curvature R is non-negative. The identity $R = (4(n-1))/n\ (C_\psi^2 + Q_\psi)$ is proved in [3]. ■

PROPOSITION 2.2. *Let (M^n, g) be a n-dimensional Riemannian spin manifold, $n \geq 3$. Suppose that (M^n, g) is an Einstein manifold with non-vanishing scalar curvature $R \neq 0$. Then*

(i) If $R > 0$ then any twistor spinor is the sum of two real Killing spinors.

(ii) If $R < 0$ then any twistor spinor is the sum of two imaginary Killing spinors.

I.e. $Ker\ \mathscr{D} = K_+ \oplus K_-$ where

$$K_+ = \left\{\psi \in \Gamma(S) : \nabla^S_X \psi = \frac{1}{2}\sqrt{\frac{R}{n(n-1)}}\ X \cdot \psi\right.$$

for all vector fields X} and

$$K_- = \left\{\psi \in \Gamma(S) : \nabla^S_X \psi = -\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}\ X \cdot \psi\right.$$

for all vector fields X.}

Proof. Let ψ be a non-trivial twistor spinor on M^n. We consider

$$\psi_1 = \frac{1}{2}\sqrt{\frac{Rn}{n-1}}\ \psi + D\psi$$

and

$$\psi_2 = -\frac{1}{2}\sqrt{\frac{Rn}{n-1}}\ \psi + D\psi.$$

Using formula (1.2), we see

$$\nabla^S_X \psi_1 = -\frac{1}{2}\sqrt{\frac{R}{n(n-1)}}\, X \cdot \psi_1$$

and

$$\nabla^S_X \psi_2 = \frac{1}{2}\sqrt{\frac{R}{n(n-1)}}\, X \cdot \psi_2$$

for all vector fields X.

Hence ψ_1 and ψ_2 are real Killing spinors, if $R > 0$ and imaginary Killing spinors, if $R < 0$.

On the other hand, we have

$$\psi = \sqrt{\frac{n-1}{Rn}}\,(\psi_1 - \psi_2). \qquad \blacksquare$$

LEMMA 2.3. *Let (M^n, g) be a n-dimensional Riemannian spin manifold, $n \geqslant 3$, with a twistor spinor ψ, $|\psi| \equiv 1$. If (M^n, g) is a Ricci-flat space, i.e. an Einstein manifold with vanishing scalar curvature $R = 0$, then ψ is a parallel spinor field.*

Proof. As in the proof of Proposition 2.1 we have $|D\psi|^2 = Rn(4(n-1))$. Since $R = 0$, this implies $D\psi \equiv 0$.

We conclude $\nabla^S_X \psi = -\,1/n\, X \cdot D\psi = 0$ for all $X \in \Gamma(TM^n)$.

Consequently, ψ is a parallel spinor field. $\blacksquare$

In the following we denote by N_ψ the zero set of the spinor ψ.

COROLLARY 2.4. *Let (M^n, g) be a n-dimensional Riemannian spin manifold, $n \leqslant 3$, with a non-trivial twistor spinor ψ and set $\bar{g} = 1/|\psi|^4 g$.*

Then $(M^n \setminus N_\psi, \bar{g})$ is an Einstein manifold with non-negative scalar curvature

$$\bar{R} = \frac{4(n-1)}{n}\,(C^2_\psi + Q_\psi).$$

If $C^2_\psi + Q_\psi > 0$, then $N_\psi = 0$ and $Ker\ \mathscr{D}$ transformas into $Ker\ \bar{\mathscr{D}}$, where $Ker\ \bar{\mathscr{D}} = \bar{K}_+ \oplus \bar{K}_-$. If $C^2_\psi + Q_\psi = 0$, then $1/|\psi|\ \bar{\psi}$ is a parallel spinor field on $(M^n \setminus N_\psi, \bar{g})$.

Proof: The Riemannian metric $\bar{g} = 1/|\psi|^4\, g$ has constant and non-negative scalar curvature

$$\bar{R} = \frac{4(n-1)}{n} (C_\psi^2 + Q_\psi) \quad \text{(see Theorem 1 in [3]).}$$

Furthermore, $1/|\psi|\ \bar{\psi}$ is a unit twistor spinor with respect to the metric $\bar{g}$. The relation $N_\psi = \phi$ for $C_\psi^2 + Q_\psi > 0$ is obviously. Now the assertion follows from Proposition 2.1, 2.2 and Lemma 2.3.

3. TWISTOR SPINORS ON 4-MANIFOLDS

Let (M^4, g) be a 4-dimensional oriented Riemannian spin manifold. Because M^4 is even-dimensional, the spinor bundle S splits into two orthogonal subbundles $S = S^+ \oplus S^-$ corresponding to the irreducible components of the Spin (4)-representation. Denote by $\psi = \psi^+ + \psi^-$ the induced decomposition of a spinor field $\psi \in \Gamma(S)$. Let W be the Weyl tensor of the Riemannian manifold M^4, which we will consider as a 2-form with values in the bundle $End(S)$ (see [3]). Denote by W_+ and W_- the components of W corresponding to the decomposition $S = S^+ \oplus S^-$.

Now suppose that $\psi = \psi^+ + \psi^-$ is a twistor spinor. Then ψ^+ and ψ^- are twistor spinors too. Furthermore, recall that $W\psi = 0$. Thus, $\psi^- \equiv 0$ implies $W_- \equiv 0$, and analogously $\psi^+ \equiv 0$ forces $W_+ \equiv 0$. Especially, if we have twistor spinors in $\Gamma(S^+)$ as well as in $\Gamma(S^-)$, then the Riemannian manifold (M^4, g) is locally conformally flat.

Furthermore, we know that the complex dimension of $Ker\ \mathcal{D}$ for a connected and simply connected Riemannian spin manifold (M^4, g) with $W \equiv 0$ is 8. Moreover, $\dim_{\mathbb{C}} Ker\ \mathcal{D} \leqslant 8$ holds on a connected Riemannian 4-manifold (see [3]). In this section we will derive further informations concerning the dimension of Ker $\mathcal{D}$ on connected and simply connected Riemannian 4-manifolds.

PROPOSITION 3.1. *If (M^4, g) is a 4-dimensional connected and simply connected Riemannian spin manifold, then the following conditions are equivalent;*

(i) $\dim_{\mathbb{C}} Ker\ \mathcal{D} \geqslant 3$

(ii) $\dim_{\mathbb{C}} Ker\ \mathcal{D} = 8$

(iii) $W \equiv 0$.

Proof: It is sufficient to show that $\dim_{\mathbb{C}} Ker\ \mathcal{D} \geqslant 3$ implies $W \equiv 0$. Let ψ_1, ψ_2, ψ_3 be three linearly independent twistor spinors. Without loss of generality we assume that $\psi_1, \psi_2, \psi_3 \in \Gamma(S^-)$. Hence $W_- \equiv 0$. On the dense subset $M^4 = M^4 \setminus N_{\psi_1}$ of M^4 we consider the metric $\bar{g} = 1/|\psi|^4 g$. Then $(\bar{M}^4, \bar{g})$ is Ricci-flat and $\bar{\varphi}_1 = 1/|\psi_1|\ \bar{\psi}_1$ is a parallel spinor. Thus $\bar{D}\bar{\varphi}_1 \equiv 0$, where $\bar{D}$ denotes the Dirac operator of the Riemannian spin manifold $(\bar{M}^4, \bar{g})$. Furthermore,

$\bar\varphi_2 = 1/|\psi_1|\ \bar\psi_2$ and $\bar\varphi_3 = 1/|\psi_1|\ \bar\psi_3$ are twistor spinors on $\bar M^4$ and $\bar\varphi_1, \bar\varphi_2, \bar\varphi_3 \in \Gamma(S^-)$ are linearly independent. Since $(\bar M^4, \bar g)$ is Ricci-flat, $\bar D\bar\varphi_2$ and $\bar D\bar\varphi_3$ are parallel spinor fields.

Suppose $\bar D\bar\varphi_2 \equiv \bar D\bar\varphi_3 \equiv 0$. Then $\bar\varphi_1, \bar\varphi_2, \bar\varphi_3$ are three linearly independent parallel spinors in $\Gamma(S^-)$. This is a contradiction to the fact that we have at most two linearly independent parallel spinors in $\Gamma(S^-)$ on the connected 4-dimensional Riemannian spin manifold $(\bar M^4, \bar g)$. Therefore, we can assume that $\bar D\bar\varphi_2 \not\equiv 0$. Because $(\bar M^4, \bar g)$ is an Einstein manifold $\bar D\bar\varphi_2 \in \Gamma(S^+)$ is a twistor spinor too. Thus $\bar W_+ \equiv 0$. By the conformal invariance of the Weyl tensor we obtain $W \equiv 0$ on a dense subset of M^4. Hence the Weyl tensor vanishes on M^4. ■

PROPOSITION 3.2. *If (M^4, g) is a 4-dimensional connected and simply connected Riemannian spin manifold, then the following conditions are equivalent:*

(i) $1 \leqslant \dim_{\mathbb{C}} Ker\,\mathscr{D} \leqslant 2$

(ii) $\dim_{\mathbb{C}} Ker\,\mathscr{D} = 2$

If one of these conditions holds and $W_- \equiv 0$ ($W_+ \equiv 0$), then we have $W_+ \not\equiv 0$ ($W_- \not\equiv 0$).

Proof: Let $\psi \not\equiv 0$ be a twistor spinor on M^4 and $W \not\equiv 0$. Without loss of generality we may assume that $\psi \in \Gamma(S^-)$. This implies $W_- \not\equiv 0$ and hence $W_+ \not\equiv 0$. On $\bar M^4 = M^4 \setminus N_\psi$ we again consider the metric $\bar g = 1/|\psi|^4\, g$. Then $\bar\varphi = 1/|\psi|\ \bar\psi$ is a parallel spinor and the curvature tensor of the Riemannian manifold $(\bar M^4, \bar g)$ has the form (see [4])

$$\bar{\mathscr{R}} = \begin{pmatrix} W_+ & 0 \\ 0 & 0 \end{pmatrix}.$$

Considering the curvature tensor $\bar{\mathscr{R}}$ as a 3-form with values in $End(\bar S)$, the curvature tensor $\bar{\mathscr{R}}^{\bar S}$ of the covariant derivative $\nabla^{\bar S}$ on $\bar S$ is given by

$$\bar{\mathscr{R}}^{\bar S}\varphi = \frac{1}{4}\ \bar{\mathscr{R}}\varphi \qquad \text{for } \varphi \in \Gamma(\bar S).$$

Thus we have $\bar{\mathscr{R}}^{\bar S}|_{\bar S^-} \equiv 0$. Hence there is a parallel spinor field $\bar\varphi_1 \in \Gamma(\bar S^-)$ with $|\bar\varphi_1| \equiv 1$ and $\langle \bar\varphi, \bar\varphi_1 \rangle \equiv 0$. It is easy to check than $\psi_1 \in \Gamma(S^-)$, defined by

$$\psi_1(x) = |\psi(x)|\ \varphi_1(x) \qquad \text{for } x \in \bar M^4 \text{ and}$$

$$\psi_1(x) = 0 \qquad \text{for } x \in N_\psi$$

is a second twistor spinor on M^4. ■

Examples

We have $\dim_{\mathbb{C}} Ker\ \mathcal{D} = 8$ for conformally flat 4-dimensional Riemannian spin manifolds (e.g. the Euclidean space $\mathbb{R}^4$ and the hyperbolic space H^4).

There are two parallel spinors in $\Gamma(S^+)$ for $K3$-surfaces. Hence, $\dim_{\mathbb{C}} Ker\ \mathcal{D} = 2$ holds for a 4-manifold which is conformally equivalent to a $K3$-surface.

REMARK In addition to the Weyl tensor we have the conformally invariant Bach tensor on a 4-dimensional oriented Riemannian manifold. A lengthy computation shows that the Bach tensor on a 4-dimensional Riemannian spin manifold with non-trivial twistor spinors vanishes identically.

4. COMPLETE CONNECTED EINSTEIN MANIFOLDS WITH NON-POSITIVE SCALAR CURVATURE ADMITTING TWISTOR SPINORS

In this section we will prove

PROPOSITION 4. *Let (M^n, g), $n \geqslant 3$, be a complete connected spin manifold. Furthermore, let (M^n, g) be an Einstein manifold with non-positive scalar curvature $R \leqslant 0$. Suppose that ψ is a non-parallel twistor spinor on M^n such that the function $f: M^n \to [0, \infty)$ defined by $f(x) = (\psi(x), \psi(x)), x \in M^n$, attains a minimum. Then*

(i) If $R < 0$, then (M^n, g) is isometric to the hyperbolic space with sectional curvature $R/(n(n-1))$.

(ii) If $R = 0$, then (M^n, g) is isometric to the space $\mathbb{R}^n$ with the standard metric.

Proof; First we consider the critical points of the function f. Clearly, $x \in M^n$ is a critical point of f if and only if $X(f) = 2/n\ (D\psi, X \cdot \psi) = 0$ for all $X \in T_x M^n$. The Hessian of f at a critical point $x \in M^n$ is given by

$$Hess_x f(X, Y) = \left[\frac{2}{n^2} \mid D\psi \mid^2 - \frac{R}{2n(n-1)} \mid \psi \mid^2\right] g(X, Y), \qquad X, Y \in T_x M^n .$$

It is known (see [3]) that if $\psi \not\equiv 0$ is a twistor spinor on M^n, then ψ and $D\psi$ do not vanish simultaneously. Thus, $R < 0$ implies that $Hess_x f$ is positive definite.

Now suppose that $R = 0$. By means of $\nabla^S_X (D\psi) = n/2\, L(X) \cdot \psi = 0$ we obtain that $D\psi$ is a parallel spinor field. Hence $\mid D\psi \mid^2$ is constant. Because ψ is non-parallel, $\mid D\psi \mid^2 > 0$ holds, which yields that $Hess_x f$ is positive definite also in the case $R \leq 0$. This shows that each critical point of f is non-degenerate and a local minimum of f. In the following we will see that f has at most one critical point: Assume that x_1 and x_2 are critical points of f and let $d = d(x_1, x_2)$ be the geodesic distance of x_1 and x_2. Now we consider a minimal geodesic $\gamma(t)$, $t \in [0, d]$, from x_1 to x_2. For the functions $u(t) = f(\gamma(t)) = |\psi(\gamma(t))|^2$ and $v(t) = |D\psi(\gamma(t))|^2$ along the geodesic γ we deduce

$$\text{(4.1)} \qquad \begin{aligned} \ddot{u} &= \frac{2}{n^2} \cdot v - \frac{R}{2n(n-1)} u \\ \dot{v} &= -\frac{Rn}{4(n-1)} \dot{u} \end{aligned}$$

Since x_1 and x_2 are critical points of f, we have $\dot{u}(0) = \dot{u}(d) = 0$. From the equations (4.1) we derive $\ddot{u} = -R/(n(n-1))\, u + A$, where $A \neq 0$ is a real constant. In the case $R < 0$ the conditions $\dot{u}(0) = \dot{u}(d) = 0$ force $d = 0$, i.e. $x_1 = x_2$.

For $R = 0$ we derive $v \equiv v(0)$ and $u(t) = v(0)/n^2\, t^2 + Bt + C$.

The condition $\dot{u}(0) = 0$ yields $B = 0$. Since ψ is a non-parallel spinor field, we have $v(0) \neq 0$. Thus, from $\dot{u}(d) = 0$ we obtain $d = 0$. Hence $x_1 = x_2$.

By the assumption f attains its minimum.

Let $x_0 \in M^n$ be the unique critical point of f. For $x \in M^n$ denote by $\gamma(t)$, $t \in [0, d(x_0, x)]$, a minimal geodesic from x_0 to x. Integrating the equations (4.1) along γ one obtains

$$f(x) = \left[f(x_0) - \frac{4(n-1)}{Rn} |D\psi(x_0)|^2 \right] \sinh^2 \left(\frac{1}{2} \sqrt{-\frac{R}{n(n-1)}}\, d(x_0, x) \right) + f(x_0),$$

$$|D\psi(x)|^2 = \left[|D\psi(x_0)|^2 - \frac{Rn}{4(n-1)} f(x_0) \right] \cosh^2 \left(\frac{1}{2} \sqrt{-\frac{R}{n(n-1)}}\, d(x_0, x) \right) + \frac{Rn}{4(n-1)} f(x_0),$$

for $R < 0$, and

$$f(x) = \frac{|D\psi(x_0)|^2}{n^2} d(x_0, x)^2 + f(x_0),$$

$$|D\psi(x)|^2 \equiv |D\psi(x_0)|^2 > 0, \text{ for } R = 0.$$

Therefore, the exponential map $exp_{x_0} : T_{x_0} M^n \cong \mathbb{R}^n \to M^n$ is a diffeomorphism and the geodesic spheres $S^{n-1}(x_0, r)$ around x_0 with radius $r > 0$ are the level surfaces of f, which are $(n-1)$-dimensional submanifolds of M^n.

Now we are going to calculate the pull back $\hat{g} = exp^*_{x_0}(g)$ of the metric g. We denote by ξ the vector field defined by

$$\xi(x) = \frac{grad\, f(x)}{\| grad\, f(x) \|}, \quad x \neq x_0 .$$

We compute

$$\nabla_X (grad\, u) = \left[\frac{2}{n^2} v - \frac{R}{2n(n-1)} u \right] X$$

for any vector field X and conclude

$$\nabla_X \xi = \frac{\left[\frac{2}{n^2} v - \frac{R}{2n(n-1)} u \right]}{\| grad\, u \|} \{ X - g(X, \xi)\xi \}.$$

This implies

(4.2) $$\nabla_\xi \xi = 0$$

Recalling that

$$C^2_\psi + Q_\psi = |\psi|^2 |D\psi|^2 - \sum_{j=1}^{n} (D\psi, e_j \cdot \psi)^2$$

is a constant and using that x_0 is a critical point of f, one obtains $C^2_\psi + Q_\psi = u(x_0)\, v(x_0)$.

Since

$$\| grad\, u \|^2 = \frac{4}{n^2} (uv - C^2_\psi - Q_\psi),$$

we arrive at

$$\| grad\, u \|^2 = \frac{4}{n^2} (uv - u(x_0)\, v(x_0)).$$

For $R < 0$ a simple calculation shows

$$\frac{2}{n^2} v(x) - \frac{R}{2n(n-1)} u(x) =$$

$$= \left[\frac{2}{n^2} v(x_0) - \frac{R}{2n(n-1)} u(x_0)\right] \cosh\left(\sqrt{\frac{R}{n(n-1)}}\, d(x_0, x)\right);$$

consequently,

$$\nabla_X \xi = \sqrt{\frac{R}{n(n-1)}} \coth\left(\sqrt{\frac{R}{n(n-1)}}\, d(x_0, x)\right) X \tag{4.3}$$

holds for all vectors $X \in T_x M^n$, $x \neq x_0$, orthogonal to $\xi(x)$. In the case $R = 0$ a similar calculation shows

$$\nabla_X \xi(x) = \frac{1}{d(x_0, x)} X \tag{4.4}$$

for all vectors $X \in T_x M^n$, $x \neq x_0$, orthogonal to $\xi(x)$.

We denote by $\gamma_t(x)$ the integral curves of ξ satisfying the condition $\gamma_0(x) = x$. Let $\Psi : S^{n-1}(x_0, 1) \times (0, \infty) \to M^n \setminus x_0$ be the diffeomorphism given by $\Psi(x, t) = \gamma_{t-1}(x)$. Using the formula (4.2), (4.3) and (4.4), we compute

$$\Psi^*(g) = \frac{\sinh^2\left(\sqrt{\frac{R}{n(n-1)}}\, t\right)}{\sinh^2\left(\sqrt{\frac{-R}{n(n-1)}}\right)}\, g|_{S^{n-1}(x_0, 1)} \oplus dt^2, \text{ if } R < 0,$$

and

$$\Psi^*(g) = t^2 g|_{S^{n-1}(x_0, 1)} \oplus dt^2, \text{ if } R = 0.$$

Applying the same arguments as in the proof of Theorem 2 in [2], one obtains that (M^n, g) is isometric to $(\mathbb{R}^n, \hat{g})$, where $\hat{g}$ is given in polar coordinates by

$$\hat{g} = -\frac{n(n-1)}{R} \sinh^2\left(\sqrt{\frac{-R}{n(n-1)}}\, t\right) g_{S^{n-1}} \oplus dt^2,$$

if $R < 0$, and

$$\hat{g} = t^2 g_{S^{n-1}} \oplus dt^2, \text{ if } R = 0.$$

Here $g_{S^{n-1}}$ denotes the standard metric of the unite sphere S^{n-1}. ■

5. THE TWISTOR EQUATION ON WARPED PRODUCTS

Let (M^{2n}, g), $n \geq 2$, be an Einstein manifold with scalar curvature $R \neq 0$.

Then the spinor bundle S of M^{2n} splits into two orthogonal subbundles $S = S^+ \oplus S^-$. Denote by $Ker\,\mathscr{D} = (Ker\,\mathscr{D})^+ \oplus (Ker\,\mathscr{D})^-$ the induced decomposition of $Ker\,\mathscr{D}$. Since (M^{2n}, g) is an Einstein manifold, we have

$$D((Ker\,\mathscr{D})^\pm) = (Ker\,\mathscr{D})^\mp .$$

Let $\{\psi_j^+\}$ be a basis of $(Ker\,\mathscr{D})^+$ and $\{\psi_j^-\}$ a basis of $(Ker\,\mathscr{D})^-$. Thus

$$D(\psi_j^+) = \sum_k D_{jk}^+ \psi_k^- \text{ and } D(\psi_j^-) = \sum_e D_{je}^- \psi_e^+ .$$

Now fix a function $f : \mathbb{R}^1 \to (0, \infty)$ and consider the Riemannian manifold $(M^{2n} \times \mathbb{R}\, f(t)^2 g \oplus dt^2)$. The metric $f(t)^2 g \oplus dt^2$ is conformally equivalent to the metric $g \oplus (f^{-1} dt)^2$. We recall that ψ is a twistor spinor on $M^{2n} \times \mathbb{R}$ with respect to the metric $g \oplus (f^{-1}dt)^2$ if and only if $\sqrt{f}\,\psi$ is a twistor spinor with respect to the metric $f(t)^2 g \oplus dt^2$.

We first consider the metric $g \oplus (f^{-1}dt)^2$ on $M^{2n} \times \mathbb{R}$.

Then $f(\partial/\partial t)$ is a normal unit vector field on M^{2n}.

Identifying $M^{2n} \times \{t\} \cong M^{2n}$ for $t \in \mathbb{R}$, we choose the spin structure of $M^{2n} \times \mathbb{R}$ so that

$$S|_{M^{2n} \times \{t\}} \cong S = S^+ \oplus S^-, \; t \in \mathbb{R},$$

for the spinor bundle S of $M^{2n} \times \mathbb{R}$, where $f(\partial/\partial t)$ acts on S by

$$f\left.\frac{\partial}{\partial t}\right|_{S^+} = i(-1)^n \text{ and } f\left.\frac{\partial}{\partial t}\right|_{S^-} = -i(-1)^n$$

(see [2]).

Let $\psi \in \Gamma(S)$ be a twistor spinor on $(M^{2n} \times \mathbb{R},\, g \oplus (f^{-1}dt)^2)$. One easily shows that $\psi|_{M^{2n} \times \{t\}}$ is a twistor spinor on (M^{2n}, g) for arbitrary $t \in \mathbb{R}$. Hence, ψ has the form

$$\psi(x, t) = \sum_j C_j^+(t)\, \psi_j^+(x) + \sum_k C_k^-(t)\, \psi_k^-(x)$$

with functions $C_j^+, C_k^- : \mathbb{R} \to \mathbb{C}$.

LEMMA 5.1. *The functions* C_j^+ , C_k^- *are given by*

$$\dot{C}_j^+ = \frac{-i(-1)^n}{2nf} \sum_k C_k^- D_{kj}^-$$

$$\dot{C}_k^- = \frac{i(-1)^n}{2nf} \sum_j C_j^+ D_{jk}^+ .$$

Proof: From

$$\psi = \sum_j C_j^+ \psi_j^+ + \sum_k C_k^- \psi_k^-$$

we obtain

$$e_i \cdot \nabla^S_{e_i} \psi = \sum_j C_j^+ \, e_i \cdot \nabla^S_{e_i} \psi_j^+ + \sum_k C_k^- \, e_i \cdot \nabla^S_{e_i} \psi_k^-$$

and

$$f \frac{\partial}{\partial t} \cdot \nabla^S_{f(\partial/\partial t)} \psi = f^2 \frac{\partial}{\partial t} \nabla^S_{(\partial/\partial t)} \psi =$$

$$= i(-1)^n f \left\{ \sum_j \dot{C}_j^+ \psi_j^+ - \sum_k \dot{C}_k^- \psi_k^- \right\},$$

where $e_1, \ldots, e_{2n}$ is a local orthonormal frame of M^{2n}. Since ψ_j^+ and ψ_k^- are twistor spinors on M^{2n}, we have

$$e_i \cdot \nabla^S_{e_i} \psi_j^+ = \frac{1}{2n} D(\psi_j^+) = \frac{1}{2n} \sum_k D_{jk}^+ \psi_k^-$$

and

$$e_i \cdot \nabla^S_{e_i} \psi_k^- = \frac{1}{2n} D(\psi_k^-) = \frac{1}{2n} \sum_j D_{kj}^- \psi_j .$$

Hence we arrive at

$$e_i \cdot \nabla^S_{e_i} \psi = \frac{1}{2n} \left\{ \sum_{kj} C_j^+ D_{jk}^+ \psi_k^- + \sum_{k,j} C_k^- D_{kj}^- \psi_j^- \right\}.$$

The twistor equation for ψ implies

$$e_i \cdot \nabla^S_{e_i} \psi = f^2 \frac{\partial}{\partial t} \nabla^S_{(\partial/\partial t)} \psi, \quad i = 1, \ldots, 2n.$$

Now the desired differential equations follow. ∎

Now assume that $\psi_j^- = D(\psi_j^+)$. Then $D_{jk}^+ = \delta_{jk}$. Futher, by means of

$$D^2 \psi_j^+ = \frac{Rn}{2(2n-1)} \psi_j^+$$

we have

$$D_{kj}^- = \frac{Rn}{2(2n-1)} \delta_{kj}.$$

Consequently, the differential equations of Lemma 5.1 become

$$\dot{C}_j^+ = \frac{-i(-1)^n R}{4(2n-1)f} C_j^- \tag{5.1}$$

$$\dot{C}_j^- = \frac{i(-1)^n}{2nf} C_j^+ \tag{5.2}$$

Differentiating equation (5.1) and using equation (5.2), we obtain

$$\ddot{C}_j^+ = \frac{R}{8n(2n-1)f^2} C_j^+ - \frac{\dot{f}}{f} \dot{C}_j^+$$

We remark that the differential equation

$$\ddot{h} = c\frac{h}{f^2} - \frac{\dot{f}}{f}\dot{h}$$

for a function h on $\mathbb{R}$ with $c \in \mathbb{R}$, $c \neq 0$, and $f : \mathbb{R} \to (0, \infty)$ has the fundamental solutions

$$h_1(t) = \sin\left(\sqrt{-c}\int_0^t \frac{d\tau}{f(\tau)}\right)$$

$$h_2(t) = \cos\left(\sqrt{-c}\int_0^t \frac{d\tau}{f(\tau)}\right) \quad \text{for } c < 0,$$

and

$$h_1(t) = \sinh\left(\sqrt{c}\int_0^t \frac{d\tau}{f(\tau)}\right)$$

$$h_2(t) = \cosh\left(\sqrt{c}\int_0^t \frac{d\tau}{f(\tau)}\right) \qquad \text{for } c > 0$$

Altogether we proved

PROPOSITION 5.2. *Let (M^{2n}, g), $n \geqslant 2$, be an Einstein manifold with scalar curvature $R \neq 0$ Let $\psi_1^+, \ldots, \psi_m^+ \in (Ker\,\mathscr{D})^+$ be a basis of $(Ker\,\mathscr{D})^+$. Then all twistor spinors of the Riemannian manifold $(M^{2n} \times \mathbb{R}, f(t)^2 g + dt^2)$, with $f : \mathbb{R} \to (0, \infty)$, are given by*

$$\psi(x, t) = \sqrt{f(t)} \sum_{j=1}^{m} \{a_j h_1(t) + b_j h_2(t)\}\, \psi_j^+(x) +$$

$$+ (\sqrt{f}(t))^3 \cdot i(-1)^n \frac{4(2n-1)}{R} \sum_{j=1}^{m} \{a_j \dot{h}_1(t) + b_j \dot{h}_2(t)\} D\psi_j^+(x),$$

where $a_j, b_j \in \mathbb{C}$ are constant and

$$h_1(t) = \sin\left(\frac{1}{2}\sqrt{\frac{R}{2n(2n-1)}}\int_0^t \frac{d\tau}{f(\tau)}\right),$$

$$h_2(t) = \cos\left(\frac{1}{2}\sqrt{\frac{-R}{2n(2n-1)}}\int_0^t \frac{d\tau}{f(\tau)}\right) \qquad \text{for } R < 0, \text{ and}$$

$$h_1(t) = \sinh\left(\frac{1}{2}\sqrt{\frac{R}{2n(2n-1)}}\int_0^t \frac{d\tau}{f(\tau)}\right)$$

$$h_2(t) = \cosh\left(\frac{1}{2}\sqrt{\frac{R}{2n(2n-1)}}\int_0^t \frac{d\tau}{f(\tau)}\right) \qquad \text{for } R > 0. \qquad \blacksquare$$

COROLLARY 5.3. *Let (M^{2n}, g), $n \geqslant 2$, be an Einstein manifold with scalar curvature $R < 0$ and let $\psi_1^+, \ldots, \psi_m^+ \in \Gamma(S^+)$ be a basis of $(Ker\,\mathscr{D})^+$. Suppose that there is a point $x_0 \in M^{2n}$ for which $\psi_1^+(x_0), \ldots, \psi_m^+(x_0) \in (S^+)_{x_0}$ as well as $D\psi_1^+(x_0), \ldots, D\psi_m^+(x_0) \in (S^-)_{x_0}$ are linearly dependent.*

Choose a number $k \in \mathbb{N}$ with

$$\int_0^\infty \frac{d\tau}{f(\tau)} \geqslant 2k \sqrt{\frac{2n(2n-1)}{-R}}\, \pi$$

for a function $f : \mathbb{R} \to (0, \infty)$.

Then there is a twistor spinor on the warped product $(M^{2n} \times \mathbb{R}, f(t)^2 g \oplus dt^2)$ *which vanishes at k points.*

Proof: By the assumptions there exist non-trivial linear combinations

$$\sum_j b_j \psi_j^+(x_0) = 0 \text{ and } \sum_j a_j D\psi_j^+(x_0) = 0.$$

Now consider the twistor spinor on $M^{2n} \times \mathbb{R}$ defined by

$$\psi(x, t) = \sqrt{f(t)} \sum_j \{a_j h_1(t) + b_j h_2(t)\}\ \psi_j^+(x) +$$

$$+ (\sqrt{f}(t))^3\, i(-1)^n\, \frac{4(2n-1)}{R} \sum_j \{a_j \dot{h}_1(t) + b_j \dot{h}_2(t)\}\, D\psi_j^+(x). \quad \blacksquare$$

Let (M^{2n+1}, g), $n \geqslant 1$, be an Einstein manifold with scalar curvature $R \neq 0$. Denote by S the spinor bundle of M^{2n+1}. Let $\psi_1, \ldots, \psi_k \in \Gamma(S)$ be a basis of *Ker* $\mathscr{D}$. Since M^{2n+1} is an Einstein manifold, we have

$$D(\psi_j) = \sum_k D_{jk}\ \psi_k.$$

Using

$$D^2 \psi_j = \frac{2n+1}{8n}\ R\psi_j,$$

we obtain

$$\sum_k D_{ik}\ D_{kj} = \frac{(2n+1)R}{8n}\ \delta_{ij}.$$

Identifying $M^{2n+1} \times \{t\} \cong M^{2n+1}$, for $t \in \mathbb{R}$, we choose the spin structure so that

$$S|_{M^{2n+1} \times \{t\}} \cong S \oplus S, \quad t \in \mathbb{R},$$

for the spinor bundle S of $(M^{2n+1} \times \mathbb{R}, g \oplus (f^{-1}dt)^2)$, where the normal unit vector field $f\,\partial/\partial t$ acts on $S \oplus S$ by $f\,\partial/\partial t\,(\varphi_1, \varphi_2) = i(-1)^n(\varphi_2, \varphi_1)$ (cf. [2]).

Now let $\psi \in \Gamma(S)$ be a twistor spinor on $(M^{2n+1} \times \mathbb{R}, g \oplus (f^{-1}dt)^2)$. Because of $\psi|_{M^{2n+1} \times \{t\}} = (\varphi_1, \varphi_2)$, where φ_1 and φ_2 are twistor spinors on (M^{2n+1}, g), ψ is described by

$$\psi(x, t) = \sum_{j=1}^{k} (A_j(t)\,\psi_j(x),\ B_j(t)\,\psi_j(x))$$

with functions $A_j, B_j : \mathbb{R} \to \mathbb{C}$.

LEMMA 5.4. *The functions A_j, B_j are given by*

$$\dot{A}_j = \frac{i(-1)^n}{(2n+1)f} \sum_k B_k D_{kj}$$

$$\dot{B}_j = \frac{i(-1)^n}{(2n+1)f} \sum_k A_k D_{kj}.$$

Proof: We have

$$e_i \cdot \nabla^S_{e_i}\psi = \sum_j (A_j e_i \cdot \nabla^S_{e_i}\psi_j,\ B_j e_i \cdot \nabla^S_{e_i}\psi_j)$$

and

$$f\frac{\partial}{\partial t} \cdot \nabla^S_{f(\partial/\partial t)}\psi = f^2\,\frac{\partial}{\partial t} \cdot \nabla^S_{(\partial/\partial t)}\psi =$$

$$= i(-1)^n f \sum_j (\dot{B}_j\,\psi_j,\ \dot{A}_j\psi_j),$$

where $e_1, \ldots, e_{2n+1}$ is a local orthonormal frame of M^{2n+1}. From $\psi_j \in Ker\,\mathscr{D}$ we deduce

$$e_i \cdot \nabla^S_{e_i}\psi = \frac{1}{2n+1} \sum_{j,k} (A_j D_{jk}\,\psi_k,\ -B_j D_{jk}\,\psi_k).$$

Applying

$$e_i \cdot \nabla^S_{e_i} \psi = f^2 \frac{\partial}{\partial t} \cdot \nabla^S_{(\partial/\partial t)} \psi$$

we obtain the assertion. ■

Differentiating the equations of Lemma 5.4, we see

$$\ddot{A}_j = \frac{R}{8n(2n+1)f^2} A_j - \frac{\dot{f}}{f} \dot{A}_j$$

and

$$\ddot{B}_j = \frac{R}{8n(2n+1)f^2} B_j - \frac{\dot{f}}{f} \dot{B}_j.$$

Altogether we have

PROPOSITION 5.5. *Let (M^{2n+1}, g), $n \geqslant 1$, be an Einstein manifold with scalar curvature $R \neq 0$. Let $\psi_1, \ldots, \psi_k \in Ker\, \mathscr{D}$ be a basis of $Ker\, \mathscr{D}$. Then all twistor spinors of the warped product*

$$(M^{2n+1} \times \mathbb{R}, f(t)^2 g \oplus dt^2), \quad f : \mathbb{R} \to (0, \infty),$$

are given by

$$\psi(x, t) = \sqrt{f(t)} \sum_{j=1}^{k} ((a_j h_1(t) + b_j h_2(t)) \psi_j(x),$$

$$(c_j h_1(t) + d_j h_2(t)) \psi_j(x))$$

where $a_j, b_j, c_j, d_j \in \mathbb{C}$ are constants coupled by Lemma 5.4, and

$$h_1(t) = \sin\left(\frac{1}{2}\sqrt{\frac{-R}{2n(2n+1)}} \int_0^t \frac{d\tau}{f(\tau)}\right),$$

$$h_2(t) = \cos\left(\frac{1}{2}\sqrt{\frac{-R}{2n(2n+1)}} \int_0^t \frac{d\tau}{f(\tau)}\right) \quad \text{for } R < 0, \text{ and}$$

$$h_1(t) = \sinh\left(\frac{1}{2}\sqrt{\frac{R}{2n(2n+1)}} \int_0^t \frac{d\tau}{f(\tau)}\right),$$

$$h_2(t) = \cosh\left(\frac{1}{2}\sqrt{\frac{R}{2n(2n+1)}} \int_0^t \frac{d\tau}{f(\tau)}\right) \quad \text{for } R > 0.$$

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Manuscript received: January 26, 1990.